# Mathematical Analysis Of Marketing Management

Julien Amédée RABOANARY, Heriniaina Andry RABOANARY, Toky Hajatiana RABOANARY, Tantely Mahefatiana RABOANARY
Department of Computer Science
Institut Supérieur Polytechnique de Madagascar
Antananarivo - Madagascar
{julien.raboanary, andry.raboanary, toky.raboanary, tantely.raboanary}@ispm-edu.com

*Abstract*—**This paper is about Marketing Management, more precisely about mathematical analysis of advertising model. This mathematical analysis concerns repeated games that involve dynamical systems. As mathematical tools, we use the concept of dynamic logic and that of control theory.**

**Keywords—Marketing management; advertising; differential dynamic logic program; differential equation; control theory; game**

## I. Introduction

This paper is about Marketing Management. More precisely it is about Mathematical analysis of advertising model. Many related researches have been done by several authors such as [1] [2] in that topic. They searched for solution of the dynamical system essentially established by Vidale [2]

In this paper, our mathematical analysis is concerned with repeated games that involve dynamical systems. As mathematical tools, we use the concept of dynamic logic and that of control theory.

More precisely the plan of the work is as follows : in section II, we introduce the concept of Differential Dynamic Logic, in section III we give the Pontrjagin principle which is a method to solve problems in control theory, in section IV we present our advertising model, in section V we solve our Differential Dynamic Logic Problems using the Pontrjagin principle, finally in section VI we analyse our marketing problem as a repeated game.

## II. Differential Dynamic Logic $dL$

**Definition 1 :**

Dynamic Logic Programs are defined as follows [3] [4]

If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are base programs then $(\boldsymbol{\alpha};\boldsymbol{\beta})$, $(\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, $\boldsymbol{\alpha}^*$ are programs in dynamic logic. We interpret the operators as :

| Proram in $d$**L** | Meaning |
|---|---|
| $\boldsymbol{\alpha};\boldsymbol{\beta}$ | Executes $\boldsymbol{\alpha}$ and then $\boldsymbol{\beta}$ such that Input of $\boldsymbol{\beta}$ is the Output of $\boldsymbol{\alpha}$ |
| $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ | Executes $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ (non-determinism) |
| $\boldsymbol{\alpha}^*$ | Executes $\boldsymbol{\alpha}$ in a finite number of times |

**Definition 2:**

A program is a *differential dynamic logic* if each one of the base program is a dynamical system. We denote by $d$**L** the set of differential dynamic logic programs.

**Remark**

In our case a base program will be one of the following :

− a differential equation :

$$\begin{cases} \dfrac{dx}{dt} = f(x,t) \\ x(0) = x_0 \end{cases}$$

- Or a problem of control theory :

$$\begin{cases} \text{Max } \mathbf{J} = \displaystyle\int_0^T v(x,u,t)dt \\ \dfrac{dx}{dt} = f(x,u,t) \\ x(\theta) = x_0 \end{cases}$$

Where **J** is the objective function.

**Definition 3 :**

If $\boldsymbol{\alpha}$ is a differential dynamic logic program then p $(\boldsymbol{\alpha})$ is the set of states reached by $\boldsymbol{\alpha}$**.**

## III. The Pontrjagin Principle

In order to solve our marketing management problem we shall introduce the Prontrjagin principle. It is a method from functional analysis which solves problems from control theory [5].

Now consider the following control theory problem

$$\begin{cases} \text{Max } \mathbf{J} = \int_0^T v(x,u)dt \\ \text{subject to } \frac{dx}{dt} = f(x,u) \\ x(0) = x^0 \quad x(T) = x^T \end{cases}$$

$x(t)$ is a state variable, **J** is the objective function $u(t)$ is continuous trajectory $0 \leq t \leq T$ $u(t)$ is the control.

We define the Hamiltonian funcion as $H(x,u,t) = v(x,u,t) + \lambda f(x,u)$.

By definition the function $\lambda$ is the costate.

The solution is found by solving [6] [7]

(i) $\frac{\partial H}{\partial u} = 0$ $\quad 0 \leq t \leq T$

(ii) $\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x}$ $\quad 0 \leq t \leq T$

(iii) $\frac{dx}{dt} = \frac{\partial H}{\partial \lambda} = f(x,u,t)$

(iv) $x(0) = x_0$

(v) $\lambda(t) = 0$ $\quad x(T) = x_T$

## IV. THE ADVERTISING MODEL

Let x(t) be the number of purchased units, $u(t)$ be the advertising rate, $c(u)$ be the advertising cost, $\pi(x)$ be the general profits. Following several authors, we assume that $c(u)$ is a particular convex cost function $c(u) = \frac{u^2}{2}$, and $\pi(x)$ a continuously differentiable revenue function.

The decision maker or manager has two *strategies* :

a) Either he decides to make advertising at the rate $u$, then his decision is designed by the following dynamical system.

**α**: **begin**

Max $\mathbf{J} = \int_0^T \left(\pi(x) - \frac{u^2}{2}\right) dt$

subject to $\dot{x} = -\delta x + u$ $\quad \dot{x} = \frac{dx}{dt}$ (1)

$x(0) = x_0$

**end**

b) Or he decides to do nothing (*i.e.* no advertising). The corresponding dynamical system is

**β** : **begin**

$$\begin{cases} \dot{x} = -\delta x \\ x(0) = x_0 \end{cases} \qquad (2)$$

**end ;**

By definition, **α** and **β** are differential dynamic programs.

## V. RESOLUTION OF $\alpha$ AND $\beta$

a) The program **β** is easy to solve

$$x(t) = x_0 e^{-\delta t}$$

b) To solve the program **α**, we_apply the Pontrjagin principle

$$H(x,u,t) = (\pi(x) - \frac{u^2}{2}) + \lambda(-\delta x + u)$$

$\frac{\partial H}{\partial u} = 0$ then $-u + \lambda = 0$

$\Rightarrow$ $u = \lambda$

$\dot{\lambda} = -\frac{\partial H}{\partial x}$ $\Rightarrow$ $\dot{\lambda} = -(\pi'(x) - \lambda)$

$\dot{x} = \frac{\partial H}{\partial \lambda} = -x + u$

$$\begin{cases} \frac{dx}{dt} = -\delta x + u \\ \frac{du}{dt} = -\pi'(x) + u \end{cases} \qquad (3)$$

We assume that at initial time $t = 0$ there is an advertising rate $u_0$ is a nonlinear system of differential equation. We look for the equilibrium states [8]:

$$\left.\begin{matrix} \dot{x} = 0 & u^* = x^* \\ \dot{u} = 0 & u^* = \pi'(x^*) \end{matrix}\right\}$$

Let $x^*, u^*$ be the solution. We have the Jacobi matrix

$$A = \begin{pmatrix} -\delta & 1 \\ -\pi''(x^*) & 1 \end{pmatrix}$$

The corresponding egeinvalues $\mu_1$ and $\mu_2$ are :

$\mu_1 = \sqrt{-\pi''(x^*) + \delta^2}$ $\quad \mu_2 = -\sqrt{-\pi''(x^*) + \delta^2}$

## VI. ANALYSIS OF THE STRATEGIES ($\alpha$; $\beta$ ) AND ($\alpha$; $\beta$ )

It is not reasonable that the manager repeat continuously without stopping any one of the two strategies **α** or **β.** This is clear for the strategy **β.** For the strategy **α,** this strategy must be stopped from time to time otherwise the consumer will be tired,at least bored of too many advertisings : » *too much marketing kills marketing* « . In other words the manager must use the strategy (**α**;**β**) in a finite number of times, namely he must apply the strategy $(\boldsymbol{\alpha};\boldsymbol{\beta})^*$.

On the other hand the manager has the possibility of choosing the mathematical formula ( or algorithm) to compute the general profits $\pi(x)$ where $x$ is the number of purchased units ; Therefore we consider four cases :

**1) <u>Case 1</u> : The manager chooses $\boldsymbol{\pi(x)}$ such that $\boldsymbol{\pi''(x) > \delta^2}$**

<u>*Analysis of the strategy* α, β, (α; β) and (α; β)*</u>

We have $-\pi''(x^*) + \delta^2 < 0$

Then the eigenvalues are

$\mu_1 = i\omega \mu_2 = -i\omega$ where $\omega = \delta^2 - \pi''(x)$

The solution in the neighborhood of equilibrium is

$$\begin{cases} x(t) = A\cos(\omega t) + B\sin(\omega t) \\ u(t) = C\cos(\omega t) + D\ \sin\omega\ t \end{cases}$$

Therefore $x(t)$ is bounded, $\Rightarrow \exists M$ such that $|x(t)| \leq M$.

Since $x(t)$ is a sinusoïd, the manager must stop doing the advertising strategy **α** at time $t_1$ where $x(t)$ is maximum, because after $t_1$ $x(t)$ begins to decrease. This is the **α part** of (**α**;**β**). After $t_1$ he does nothing (it is the strategy **β** ) until time $t_2$. Hence we obtain (**α**, **β**) and this compound strategy is repeated a finite number of times as needed. As represented in Fig. 1.

**2) <u>Case 2</u> : The manager chooses $\pi(x)$ such that: $0 < \pi''(x) < \delta^2$**

<u>*Analysis of the strategy* α, β, (α; β) and (α; β)*</u>

The function $\pi(x)$ is a convex function. Let $-\pi''(x) + \delta^2 > 0$

If $\omega = \sqrt{\delta^2 - \pi''(x)}$

$$\begin{cases} x(t) = A_1 e^{-\omega t} + B_2 e^{\omega t} \\ u(t) = A_2 e^{-\omega t} + B_2 e^{\omega t} \end{cases}$$

For example

Take $\pi(x) = mx^2$to be a particular convex function, therefore $\pi''(x) = 2m$ with $< \frac{\delta^2}{2}$ ,

It is easy to see that there exists some $t_1$ such that after $t_1$ the objective function **J** becomes negative. From t=0 up to t= $t_1$ corresponds to the differential dynamic program **α**. After $t_1$ the manager must do nothing up to some time $t_2$. This corresponds to the differential dynamic program **β.** From t = 0 up to t= $t_2$we have the compound differential dynamic program (**α**;**β**)**.** This program is repeated in finite number of times and we get (**α**;**β**)*. As shown in Fig. 2.

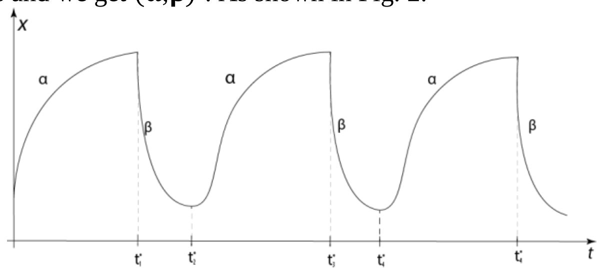


Fig. 1. : $(\alpha\,;\beta)^*$: case 1

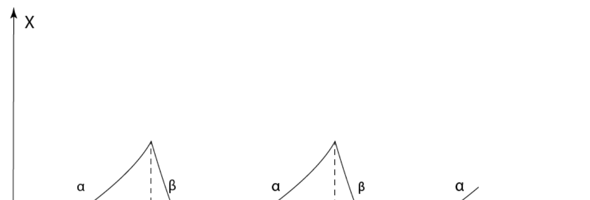


Fig. 2. : $(\alpha;\beta)^*$ : case 2

**3) <u>Case 3</u> : The manager chooses $\pi(x)$ such that: $\pi''(x) = 0$ (Fig. 3)**

<u>*Analysis of the strategy* α, β, (α; β) and (α; β)*</u>

Therefore $\pi(x) = mx$

We have a linear system. We do not need analyses about the neighborhood of the equilibria point.

The system (3) is written as $\begin{cases} \frac{dx}{dt} = -\delta x + u \\ \frac{du}{dt} = -m + u \end{cases}$

$$x(t) = x_0 e^{-\delta x} + u_0 t e^{t} - m e^{-t}$$

As in case 3, the objective function **J** becomes negative after some $t_1$ and we obtain the same conclusion as above.

**4) <u>Case 4</u> : The manager chooses $\pi(x)$ such that: $\pi''(x) < 0$ (Fig. 4)**

<u>*Analysis of the strategy* α, β, (α; β) and (α; β)*</u>

$\pi''(x) < 0$ $\pi(x)$ is a concave function. $\omega = -\pi(x) + \delta^2 > 0$

The solution is $\begin{cases} x(t) = A_1 e^{-\omega t} + B_1 e^{\omega t} \\ u(t) = A_2 e^{-\omega t} + B_2 e^{\omega t} \end{cases}$

For example

Choose a particular $\pi(x) = m(2Nx - x^2)$ we have

$\pi''(x) = -2m < 0$

$$\begin{cases} x(t) = Ae^{-\omega t} + B\, e^{-\omega t} \\ u(t) = A_2 e^{-\omega t} + B_2 e^{-\omega t} \end{cases}$$

So $x(t) = K_1\,(e^{\omega t})$, $K > 0$

$$u(t) = K_2(e^{\omega t})$$

$$\exists K\; u = Kx$$

$$\mathbf{J} = \int_0^T (\pi(x).\frac{u^2}{2})dt$$

⇒ to make sure that **J** ≥ 0

Therefore $\exists K\; 2Nx - x^2 \geq Kx^2$

$$\Rightarrow \boldsymbol{x \leq \frac{2N}{K+1} = M}$$

But $x(t) = K_1 e^{\omega t}$

$$\Rightarrow t^* = \ln(\frac{M}{K_1})$$

$$t^* = (h\left(\frac{N}{A}\right))$$

The manager applies strategy **α**, then he must stop advertising at $t^*$, because after $t_1^*$ **J** becomes negative. Therefore after $t_1^*$, the manager does nothing (strategy **β**) until $t_2^*$.

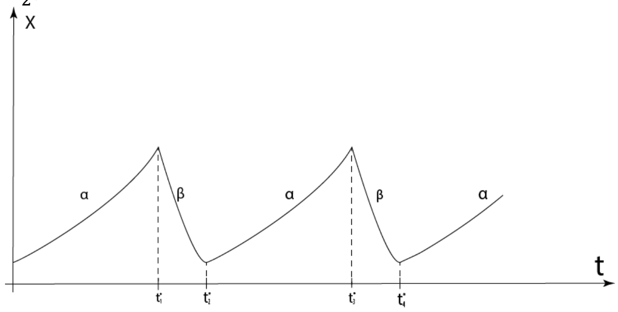


Fig. 3. : $(\alpha\,;\beta)^*$: case 3

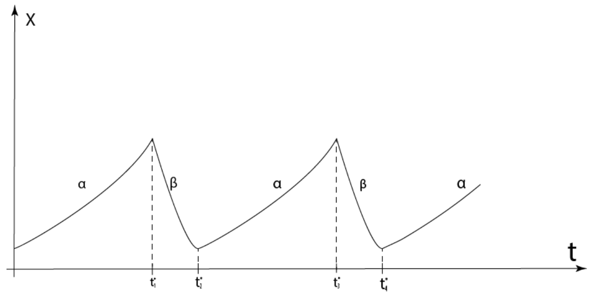


Fig. 4. $(\alpha\,;\beta)^{*}$: case 4

## VII. CONCLUSION

This paper is concerned with repeated games on advertising model. We found some results such that

1) the result in case 1 is not interesting because the number purchased units is bounded by a constant

2) the results in cases 2, 3 and 4 are more interesting because the number of purchased units is of exponential order.

In the future, we may apply our method based on differential dynamic logic to other marketing and economic problems, more specifically to other more sophisticated dynamical descending advertising models [1].

## AKNOWLEDGMENT

We thank the ISPM – Institut Supérieur Polytechnique de Madagascar – for financially supporting our work. Especially, we thank RABOANARY Narindra Lova Haingotiana, ANDRIANARAHANA Narihasina, RANOELINTSOA Elie and RAZAFITSIAROVANA Mahenina for their precious help.

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